MINIMAL MODELS AND WEAK ZARISKI DECOMPOSITIONS

$/\mathbb{C}$

var. are normal and proj.

· CONJ (Existence of minimal models)
Let (X,B) be an lc pair. If K_X+B is psef, then (X,B) has a minimal model.

that is, $\exists\, \varphi : (X,B) \dashrightarrow (Y,B_Y)$
s.t. K_Y+B_Y : nef.

- Known cases:

· $\dim X = 2$:

- X: smooth, $B=0$: classical
- general case:

· $\dim X = 3$:

'80s-'90s by the works Kawamata, Mori, Shokurov, and other.

- $\dim X = 4$:
  - X: terminal, $B = 0$: Kawamata-Matsuda-Matsuki '87
  - general case:
    - weaker versions by
      Shokurov (2009)
      Birkar (2012)
    - complete solution: Lazić-T. (2019)

- $\dim X = 5$: $\exists$ several partial results
  Birkar (2010)
  Lazić-T. (2019)

In general, the CONJ is open in higher dimensions!

However,

$$\left.\begin{array}{l}\text{Birkar-Cascini-Hacon-McKernan}\\ \text{Cascini-Corti-Lazić}\end{array}\right\} \Rightarrow \exists \text{ MM for klt pairs of general type (any dim).}$$

<u>Goal</u>: to present the tools that enable us to obtain the dim=4 case (the partial dim=5 case)

§ PRELIMINARIES

(1) generalized pairs:

for more info. see talks from

[illegible]

11.02.2022, 05.03.2022, etc.

DEF: A generalized pair (g-pair) consists of the following data:

- a normal variety X,
- an effective $\mathbb{R}$-divisor B on X,
- a proj. birat. morphism $f: X' \to X$ from a normal variety X' and an $\mathbb{R}$-Cartier M' on X' which is nef,

such that $K_X + B + M$ is $\mathbb{R}$-Cartier where $M := f_* M'$.

- notation: $(X, B+M)$
- standard assumption: M' is NQC (= nef $\mathbb{Q}$-Cartier combinations), i.e.

$M' = \sum_{j=1}^{\ell} \mu_j M_j'$, where $\mu_j \geq 0$ and

M_j' : nef $\mathbb{Q}$-Cartier divisors.

Why NQC?

- MMP behaves better [Han-Li]
- ACC for LCTs, Global ACC [Birkar-Zhang 16]
- without this assumption, some non-vanishing statements fail [Han-Liu].

· We can MMPs for NQC lc g-pairs whose underlying variety is $\mathbb{Q}$-factorial [Hacon-Liu].

(2) Minimal Models:

· <u>DEF</u>: Let $(X, B+M)$ be an lc g-pair. Assume that we have a birational map $\varphi : (X, B+M) \dashrightarrow (Y, B_Y + M_Y)$ to a g-pair $(Y, B_Y + M_Y)$, where M and M_Y are pushforwards of the same nef $\mathbb{R}$-divisor on a common birat. model of X and Y. We say that

(a) $(Y, B_Y + M_Y)$ is a minimal model (MM) in the sense of Birkar-Shokurov (BS) of $(X, B+M)$ if

- $B_Y = \varphi_* B + E$, where E is the sum of all (φ^{-1})-exceptional prime divisors on Y,
- Y: $\mathbb{Q}$-factorial
- $K_Y + B_Y + M_Y$: nef

for any φ-exceptional prime divisor F on X we have

$$a(F, X, B+M) < a(F, Y, B_Y + M_Y).$$

negativity lemma $\Rightarrow$ $(Y, B_Y + M_Y)$ is lc.

(b) $(Y, B_Y + M_Y)$ is called a minimal model (MM) in the usual sense if, additionally, φ is a birational contraction (that is, φ^{-1} does not contract any divisor $\Rightarrow E = 0$), and Y need not be $\mathbb{Q}$-factorial if X itself is not $\mathbb{Q}$-factorial.

Q: How different are these two notions?

A: - coincide for klt g-pairs

- essentially equivalent even more generally:

<u>THM</u>:

(i) [Hashizume-Hu, 2020]: If (X,B) is an lc pair, then it has a MM in the sense of BS iff it has a MM.

(ii) [Lazić-T., 2021]: If $(X,B+M)$ is an NQC $\mathbb{Q}$-fact. lc g-pair, then it has a MM in the sense of Birkar-Shokurov iff it has a MM.

§ WEAK ZARISKI DECOMPOSITIONS

Recall (Zariski Decomp.): Let X be a smooth projective surface and let D

be an effective $\mathbb{Z}$-divisor on X. Then D can be written uniquely as a sum

$$D = P + N$$

of $\mathbb{Q}$-divisors satisfying:

- P: nef
- $N = \sum a_i E_i \geq 0$, and if $N \neq 0$, then the intersection matrix $(E_i \cdot E_j)$ is negative definite.
- $P \cdot E_i = 0$ for each i.

· DEF (Birkar): Let X be a normal proj. variety and let D be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X. A NQC weak Zariski decomposition (WZD) of D consists of a proj. birational morphism

$f: W \to X$ from a normal variety W
and a numerical equivalence

$$f^*D \equiv P + N,$$

where P is nef/NQC and $N \geq 0$
$\mathbb{R}$-Cartier $\mathbb{R}$-divisor.

<u>LEM 1</u>: (Basic properties):

(i) $\left.\begin{array}{l} D: \text{admits NQC WZD} \\ g: Y \to X: \text{proj. surj.} \end{array}\right\} \Longrightarrow$

$\Longrightarrow g^*D$: admits NQC WZD.

(ii) $\left.\begin{array}{l} g: \text{birational (as above)} \\ g^*D: \text{admits NQC WZD} \end{array}\right\} \Longrightarrow$

$\Longrightarrow D$: admits NQC WZD

(iii) Assume that D admits NQC WZD.

(a) $G \geq 0$ $\mathbb{R}$-Cartier $\Rightarrow$ $D+G$ admits NQC WZD

(b) Q: $\mathbb{R}$-Cartier s.t. Q is the pushforward of an NQC divisor sitting on some higher model $\Rightarrow$

$\Rightarrow$ $D+Q$: admits NQC WZD.

(iv) X, Y: $\mathbb{Q}$-factorial, proj. var.

$f: X \dashrightarrow Y$: D-non-positive map (e.g. step of a D-MMP)

$\Longrightarrow$ (D: admits NQC WZD iff $f_* D$: admits NQC WZD)

We say that an NQC g-pair $(X, B+M)$ admits an NQC WZD if $K_X + B + M$ admits an NQC WZD.

PROP: Let $(X, B+M)$ be an NQC lc g-pair. If $(X, B+M)$ has an MM (in any sense), then $(X, B+M)$ admits NQC WZD.

PROOF (Sketch)

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \varphi : (X, B+M) \arrow[rr, dashed] & & (Y, B_Y+M_Y) : \text{MM} \end{tikzcd}$$

resolution of indet.: (p, q)

negativity lemma $\Rightarrow$ $p^*(K_X+B+M) \sim_{\mathbb{R}}$

$$\sim_{\mathbb{R}} q^*(\underline{K_Y+B_Y+M_Y}) + \underline{E},$$

where $E \geq 0$ and q-exc.

Shokurov $\Rightarrow$ $K_Y+B_Y+M_Y$: NQC.

Polytopes

Hence, $(X, B+M)$ has a NQC WZD. $\boxtimes$

Q: How about the converse?

A:

① Birkar (2012): Assuming the termination of flips in dim $n-1$, an lc pair (X,B) of dim n has MM iff it admits a WZD.

in the sense of BS.

② Han-Li (2018): Assume the existence of NQC WZD for NQC lc g-pairs of dim $\leq n-1$.

Let $(X, B+M)$ be an NQC lc g-pair of dim n which admits an NQC WZD. Then $(X, B+M)$ has MM in the sense of BS.

Furthermore, if X is $\mathbb{Q}$-factorial klt, then any (K_X+B+M)-MMP with scaling of an ample divisor terminates.

↳ has been improved significantly by Lazić-T. (2019, 2021) leading to the following

③ Lazić-T. : Assume the existence of MM for smooth varieties of dim $n-1$.

(i) If (X,B) is an lc pair of dim n, then (X,B) has a MM iff (X,B) admits NQC WZD.

(ii) If $(X,B+M)$ is an NQC $\mathbb{Q}$-factorial lc g-pair of dim n, then $(X,B+M)$

has a MM iff $(X, B+M)$ admits NQC WZD.

LEM 2: Pick $n, k \in \mathbb{Z}_{\geq 1}$ s.t. $n \geq k$. Assume every sm. proj. var. of dim n with psef canonical class admits an NQC WZD. Then every sm. proj. var. of dim k $(\leq n)$ with psef canonical class admits an NQC WZD.

THM (key result): Assume the existence of NQC WZD for NQC lc g-pairs of dim $\leq n-1$.

Let $(X, (B+N)+(M+P))$ be an NQC $\mathbb{Q}$-factorial dlt g-pair of dim n.

Assume that $K_X + B + N + M + P$ is psef and that for every $\varepsilon > 0$, $K_X + B + M + (1-\varepsilon)(N+P)$ is not psef. Then $(X, (B+N)+(M+P))$ admits an NQC WZD.

PROOF (Sketch)

By running various MMP, we may assume that there exists a fibration $g : X \to T$ s.t.

- $\boxed{0 < \dim T < \dim X}$,
- $K_X + B + M + N + P \sim_{\mathbb{R}} g^* D_T$ for some $\mathbb{R}$-Cartier divisor D_T on T.

$\overset{\text{CBF}}{\Longrightarrow} \exists\, (T, B_T + M_T)$: NQC lc g-pair such that

$(*)\ K_X + B + M + N + P \sim_{\mathbb{R}} g^*(K_T + B_T + M_T)$

$\Longrightarrow K_T + B_T + M_T$: psef

$\overset{\text{ass.}}{\Longrightarrow} (T, B_T + M_T)$: admits NQC WZD

$\overset{(*)}{\Longrightarrow} K_X + B + M + N + P$: admits NQC WZD.

LEM 1

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· <u>THM 2</u> : The existence of NQC WZD for smooth var. of dim n implies the existence of NQC WZD for NQC lc g-pairs of dim n.

<u>PROOF</u>

Fix $(X, B+M)$: NQC lc g-pair s.t. $K_X + B + M$ is psef.

· ass $\overset{\text{LEM 2}}{\Longrightarrow}$ $\exists$ NQC WZD for sm. var. of dim $\leq n$

induct...

Inductive hypothesis $\exists$ NQC WZD for NQC lc g-pairs of dim $\leq n-1$.

- By LEM 1, we may assume that $(X, B+M)$ is log smooth. Set

$$\tau := \inf\{t \geq 0 \mid K_X + t(\underline{\underline{B+M}}) \text{ : psef}\} \in [0,1].$$

- If $\tau = 0$ (i.e., K_X is psef), then by assumption X admits an NQC WZD, so by LEM 1, $K_X + B + M$ admits NQC WZD.

- If $0 < \tau \leq 1$, then by THM (key result), $K_X + \tau(B+M)$ admits NQC WZD, so $K_X + B + M$ admits NQC WZD by LEM 1. $\blacksquare$

COR 1: Assume the existence of NQC WZD for NQC lc g-pairs of dim $\le n-1$.
Let $(X, B+M)$ be an NQC lc g-pair of dim n s.t. X is uniruled and K_X+B+M is psef. Then $(X, B+M)$ admits an NQC WZD.

PROOF

Argue as before, using that
(X: uniruled iff K_X: not psef)
for X: smooth. $\blacksquare$

§ FROM WZD TO MM

THM 3: The existence of MM for

smooth varieties of dim n implies

(i) the existence of MM for lc pairs of dim n, and

(ii) the existence of MM for NQC $\mathbb{Q}$-factorial lc g-pairs of dim n.

In particular, the existence of MM conj. holds in dim ≤ 4.

THM 4: Assume the existence of MM for smooth var. of dim $n-1$.

(i) If (X,B) is an lc pair of dim n s.t. X is uniruled and K_X+B is psef, then (X,B) has MM.

(ii) If $(X,B+M)$ is an NQC $\mathbb{Q}$-factorial lc g-pair of dim n s.t. X is uniruled and K_X+B+M is psef, then $(X, B+M)$ has a MM.

In particular, the Existence of minimal models conjecture holds for such pairs/g-pairs in dim 5